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THE PERTURBATION METHOD IN PROBLEMS OF THE DYNAMICS OF INHOMOGENEOUS ELASTIC RODS†

L. D. AKULENKO and G. V. KOSTIN

Moscow

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The regular perturbation method (the small-parameter method) is developed in order to investigate the dynamics of weakly inhomogeneous rods with arbitrary distributed loads and boundary conditions of various types leading to self-conjugate boundary-value problems. The approach rests on the introduction of a perturbed argument, namely, the Euler variable, and a suitable representation of the eigenfunctions. It enables one to carry out uniform constructions of the basis and the eigenvalues, as well as the frequencies with any required accuracy in terms of the small parameter using quadratures of known functions. To illustrate the effectiveness, an example involving inhomogeneous rods with hinged left-hand ends and free right-hand ends and with box-shaped and circular cross-sections whose dimensions depend linearly on the coordinate are investigated and computed.

1. FORMULATION OF THE PROBLEM

CONTROLLED planar motions of an elastic rod undergoing transverse bending deformations are considered. Longitudinal extension will be neglected. It is assumed that the neutral line of the unstrained rod is straight and the elastic strains are small, i.e. the motion of the rod can be described

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in the framework of the linear theory of thin elastic rods [1, 2]. It is assumed that the inertial and stiffness characteristics are constant with time and the conditions of motion are such that the dynamic equation for the cross-sections (the state equation) has the form

$$\rho(x)u^{\cdot\cdot}=-(\sigma(x)u'')''+W(t,x),\quad u=u(t,x),$$
$$0<x<l,\quad t\in[0,\ T] \tag{1.1}$$

Here $u(t,x)$ is the transverse displacement of the neutral line with Euler coordinate x at time t, $W(t,x)$ is the external action, which is a known, sufficiently smooth function for all $x\in[0,l]$;, and l is the length of the rod, which is assumed to be constant. The linear density ρ and bending stiffness σ are assumed to be stationary (i.e. independent of t) sufficiently smooth functions of x that satisfy the conditions

$$0<\rho_{\min}\leqslant\rho(x)\leqslant\rho_{\max}<\infty,\ 0<\sigma_{\min}\leqslant\sigma(x)\leqslant\sigma_{\max}<\infty,\ 0\leqslant x\leqslant l \tag{1.2}$$

where the bounds of the intervals of variation are sufficiently close to one another (see below).

We adopt the standard form of the boundary conditions, i.e. the values of $u(t,x)$ at $x=0$ and $x=l$, leading to a self-conjugate boundary-value problem [3, 4]. It is assumed that the following simple inhomogeneous conditions hold for the values of $t\in[0,T]$ under consideration.

1. Clamping (rigid constant) of the left end ($x=0$) and/or the right end ($x=l$) of the rod:

$$u(t,\ x)|_{x=0,\,l}=S_{0,\,l}(t),\quad u'(t,\ x)|_{x=0,\,l}=K_{0,\,l}(t) \tag{1.3}$$

2. Free left end ($x=0$) and/or right end ($x=l$) of the rod:

$$-[\sigma(x)u''(t,\ x)]|_{x=0,\,l}=M_{0,\,l}(t),$$
$$-[\sigma(x)u''(t,\ x)]'|_{x=0,\,l}=P_{0,\,l}(t) \tag{1.4}$$

3. Hinged attachment of the left end ($x=0$) and/or the right end ($x=l$) of the rod:

$$u(t,\ x)|_{x=0,\,l}=S_{0,\,l}(t),\quad -[\sigma(x)u''(t,\ x)]|_{x=0,\,l}=M_{0,\,l}(t) \tag{1.5}$$

4. Free left end ($x=0$) and/or right end ($x=l$) of the rod with fixed tangent:

$$u'(t,\ x)|_{x=0,\,l}=K_{0,\,l}(t),\quad -[\sigma(x)u''(t,\ x)]'|_{x=0,\,l}=P_{0,\,l}(t) \tag{1.6}$$

The mathematical meaning of the sufficiently smooth functions of t ($t\in[0,T]$) introduced in (1.3)–(1.6) is clear. They characterize both the kinematic actions: $S_{0,l}(t)$ is the prescribed displacement and $K_{0,l}(t)$ is the prescribed direction of the tangent line, and the dynamic actions: $M_{0,l}(t)$ is the applied external moment of forces orthogonal to the neutral line and $P_{0,l}(t)$ is the external shear force orthogonal to that line. The above functions, as well as the distributed external action $W(t,x)$ in (1.1), can include perturbations and control actions of a kinematic or dynamic nature [1, 2, 5]. For simplicity, they are assumed to be prescribed, i.e. independent of the unknown function $u(t,x)$ to be determined or its derivatives at $x=0,l$. We remark that, on the basis of n forms of the boundary conditions (1.3)–(1.6) at one or both ends of the rod, we can state $N=n+n(n-1)/2=n(n+1)/2$ (in our case $n=4$ and $N=10$) forms the boundary conditions for the whole rod, i.e. 10 forms of distinct boundary value problems. Since the rod is inhomogeneous, n^2 different solutions are possible. Each of the corresponding boundary value problems is self-conjugate, which can be established directly on the basis of the definition [3, 4, 6, 7]. Under external actions of a certain class, the solutions of the problems will belong to the corresponding class provided the initial conditions are specified. The latter can be taken in the standard form for $t=0$:

$$u(0,\ x)=f^0(x),\quad u^{\cdot}(0,\ x)=g^0(x) \tag{1.7}$$

If the control problem is posed, then suitable terminal conditions can also be given for $t=T$:

$$u(T,\ x)=f^T(x),\quad u^{\cdot}(T,\ x)=g^T(x) \tag{1.8}$$

The functions $f^{0,T}(x)$ and $g^{0,T}(x)$ must be sufficiently smooth, or, more precisely, they must belong to a certain class of smoothness in order that the solution $u(t, x)$ to be determined exist in the required class [3, 4].

A constructive solution of the problems stated can be obtained by the method of separation of variables (the Fourier method) as an infinite sum of $u_n(t, x) = \Theta_n(t)X_n(x)$, where $n = 0, \pm 1, \pm 2, \ldots$, i.e. as a series in terms of the system of eigenfunctions $\{X_n(x)\}$, which are orthogonal with weight $\rho(x)$ and form a basis. The boundary-value problems for the eigenvalues and eigenfunctions corresponding to the boundary-value problems (1.1), (1.3)–(1.6) for $x = 0$ and/or $x = l$ have the form

$$(\sigma X'')'' - \lambda^4 \rho X = 0, \quad 0 < x < l, \quad \lambda = \text{const} \tag{1.9}$$

$$1)\ X = X' = 0,\quad 2)\ \sigma X'' = (\sigma X'')' = 0$$

$$3)\ X = \sigma X'' = 0,\quad 4)\ X' = (\sigma X'')' = 0 \quad (x = 0,\ x = l)$$

The solutions of (1.9) are known for constant ρ and σ [1–3, 5]. The eigenfunctions $X_n(x)$ can be found in the form of combinations of trigonometric and hyperbolic sine and cosine functions, and the real eigenvalues λ_n can be found as the roots of the transcendental characteristic equations. By symmetry, it is then sufficient to restrict ourselves to the values of $X_n(x)$ and λ_n for $n = 0, 1, 2, \ldots$, i.e. not to consider $n = -1, -2, \ldots$

In the more general case of an inhomogeneous rod with ρ and σ depending on x, the eigenvalues and eigenfunctions can be approximately determined using constructive algorithms of the perturbation method if $\rho(x) \approx \rho_0$ and $\sigma(x) \approx \sigma_0$, where ρ_0 and σ_0 are positive constants. To make it convenient to use the perturbation method developed below, we introduce a small numerical parameter ε such that $0 \leqslant \varepsilon \ll 1$, which characterizes this closeness. Taking (1.2) into account, we write down the identifies

$$\rho(x) = \rho_0[1 + \varepsilon\delta(x, \varepsilon)], \quad \sigma(x) = \sigma_0[1 + \varepsilon c(x, \varepsilon)] \tag{1.10}$$

$$\varepsilon\delta \equiv (\Delta\rho/\rho_0)(\rho - \rho_0)/\Delta\rho, \quad \varepsilon c \equiv (\Delta\sigma/\sigma_0)(\sigma - \sigma_0)/\Delta\sigma$$

$$\Delta\rho = (\rho_{\max} - \rho_{\min})/2, \quad \Delta\sigma = (\sigma_{\max} - \sigma_{\min})/2$$

$$\rho_0 = (\rho_{\max} + \rho_{\min})/2, \quad \sigma_0 = (\sigma_{\max} + \sigma_{\min})/2$$

Assuming, for example, that $\Delta\rho/\rho_0 \sim \varepsilon$ and $\Delta\sigma/\sigma_0 \sim \varepsilon$, where ε is the small numerical parameter such that $\varepsilon \in [0, \varepsilon_0]$ with $0 < \varepsilon_0 \ll 1$, by (1.10) we have $|\delta| \leqslant 1$ and $|c| \leqslant 1$ for $x \in [0, l]$. In the limit as $\varepsilon \to 0$ we obtain self-conjugate boundary-value problems with constant characteristics $\rho = \rho_0$ and $\sigma = \sigma_0$ of the rod, the solutions of which can be constructed in the form of quadratures on the basis of the known systems of eigenvalues $\{\lambda_n^{(0)}\}$ and eigenfunctions $\{X_n^{(0)}\}$. We can set $l = \rho_0 = \sigma_0 = 1$ in (1.10). This can be achieved by setting $x_* = x/l$ and $\lambda_* = \lambda(\rho_0/\tau_0)^{1/4}$ and then suppressing the index *. The functions X, δ and c can be transformed into functions of the new argument x_* and the parameter λ_*. As a result, we obtain a set of 10 self-conjugate boundary-value problems containing one given parameter ε such that $0 < \varepsilon \ll 1$ ($\lambda(\varepsilon)$ and $X(x, \varepsilon, \lambda)$ are to be determined):

$$((1 + \varepsilon c(x))X'')'' - \lambda^4(1 + \varepsilon\delta(x))X = 0, \quad 0 < x < l \tag{1.11}$$

$$1)\ X = X' = 0,\quad 2)\ X'' = ((1 + \varepsilon c(x))X'')' = 0$$

$$3)\ X = X'' = 0,\quad 4)\ X' = ((1 + \varepsilon c(x))X'')' = 0 \quad (x = 0,\ x = l)$$

For $\varepsilon = 0$ (the case of a homogeneous rod), the solutions of the boundary-value problems are known [1, 2, 4, 5]. On the basis of these solutions one can construct the desired solutions $u^{(0)}(t, x)$ of the original initial-value problems (1.1), (1.3)–(1.7) in t. For $\varepsilon > 0$ the problem arises of the existence of solutions of the required class of smoothness and the construction of these solutions by the methods of perturbation theory [7–9].

Thus, the problem arises of constructing the system of eigenvalues $\{\lambda_n(\varepsilon)\}$ and a complete system of functions $\{X_n(x, \varepsilon)\}$ orthogonal with weight $(1 + \varepsilon\delta(x))$ with the required degree of accuracy in ε uniform in n as $|n| \to \infty$. We remark (see [10]) that direct substitution into (1.11) of the expansions

$$\lambda_n(\varepsilon) = \lambda_n^{(0)} + \varepsilon\lambda_n^{(1)} + \ldots + \varepsilon^k\lambda_n^{(k)} + \ldots$$

$$X_n(x, \varepsilon) = X_n^{(0)}(x) + \varepsilon X_n^{(1)}(x) + \ldots + \varepsilon^k X_n^{(k)}(x) + \ldots \tag{1.12}$$

of $\varepsilon = 0$ and $X_n(x, \varepsilon)$ in terms of the powers of ε leads to "secular terms" of the form $\varepsilon^p n^q$ (p and q being natural numbers) due to the expression $\varepsilon\lambda^4\delta(x)X$, the smallness of which is doubtful for $\lambda \to \infty$. This fact is unsatisfactory both theoretically and in applications when such approximate expressions are used as a basis, since both the absolute and relative errors will increase without limit as n increases ($n \to \infty$). The class of boundary-value problems under consideration is much more difficult to study than in the case of the second-order differential equation in x considered in [10]. However, the fundamental approaches [10] leading to the regularization of the process of constructing $\{X_n(x, \varepsilon)\}$ can be applied in the case being discussed. A constructive method of regularizing perturbed boundary-value problems can be proposed. Below we describe and discuss the algorithmic aspects of the problem and some questions concerned with the justification. From the point of view of functional analysis, an additional thorough investigation of the properties of the approximate basis is needed.

2. TRANSFORMATION OF THE INDEPENDENT VARIABLE

We propose a method which involves the introduction of the perturbed argument y and parameter ν by means of transformation formulas that are close to identities [10]. For $y = y(x, \varepsilon)$, we consider the following expression:

$$\begin{gathered} y=y(x,\varepsilon)=[x+\varepsilon\varphi(x,\varepsilon)][1+\varepsilon\varphi(1,\varepsilon)]^{-1}=x+\varepsilon\xi(x,\varepsilon), \\ x=y+\varepsilon\eta(x,\varepsilon) \\ \varphi(0,\varepsilon)=\xi(0,\varepsilon)=\xi(1,\varepsilon)=\eta(0,\varepsilon)=\eta(1,\varepsilon)=0; \\ x\in[0,1],\ y\in[0,1] \\ \varphi(x,\varepsilon)=\int_0^x\theta(z,\varepsilon)\,dz,\quad \theta=\theta(x,\varepsilon)=\frac{1}{\varepsilon}\left[\left(\frac{1+\varepsilon\delta(x)}{1+\varepsilon c(x)}\right)^{1/4}-1\right]= \\ =(\delta(x)-c(x))/4+O(\varepsilon) \end{gathered} \tag{2.1}$$

For sufficiently small $\varepsilon > 0$, Eqs (2.1) define a one-to-one relation between x and y with $y = x$ for $\varepsilon = 0$. Instead of λ, which remains unknown for the time being, we introduce the following parameter:

$$\nu=\lambda(1+\varepsilon\varphi_1(\varepsilon)),\quad \varphi_1(\varepsilon)=\varphi(1,\varepsilon) \tag{2.2}$$

The unknown function X to be determined can be converted to the form

$$X=X(x,\lambda,\varepsilon)\equiv Y(y,\nu,\varepsilon)=Y \tag{2.3}$$

The differential equation (1.11) for the unknown function Y can be written as follows:

$$Y^{IV}-\nu^4 Y=\varepsilon(AY'''+BY''+CY'),\quad 0<y<1 \tag{2.4}$$

$$A=A(y,\varepsilon)\equiv-2\frac{1+\varepsilon\varphi_1}{1+\varepsilon\theta}\left[\frac{3\theta'}{1+\varepsilon\theta}+\frac{c'}{1+\varepsilon c}\right]\Bigg|_{x=y+\varepsilon\eta}$$

$$B=B(y,\varepsilon)\equiv-\left(\frac{1+\varepsilon\varphi_1}{1+\varepsilon\theta}\right)^2\left[\frac{4\theta'+3\varepsilon\theta'^2}{1+\varepsilon\theta}+\right.$$
$$\left.+\frac{6\varepsilon c'\theta'}{(1+\varepsilon\theta)(1+\varepsilon c)}+\frac{c''}{1+\varepsilon c}\right]\Bigg|_{x=y+\varepsilon\eta}$$

$$C=C(y,\varepsilon)\equiv-\frac{(1+\varepsilon\varphi_1)^3}{(1+\varepsilon\theta)^4}\left[\theta'''+\varepsilon\frac{2c'\theta''+c'\theta'}{1+\varepsilon\theta}\right]\Bigg|_{x=y+\varepsilon\eta}$$

$$\theta=\theta(x,\varepsilon),\quad x=y+\varepsilon\eta,\quad \eta=\eta(y,\varepsilon)$$

Equation (2.4) is defined if $\delta(x)$ and $c(x)$ are of class C^3 for all $x\in[0, 1]$. We note that $A = B = C = 0$ if δ and c are constant for all $x\in[0, 1]$. The boundary conditions 1–4 in (1.11) can be transformed using the relation between x and y on the basis of the expressions for X and Y and their derivatives:

$$X(x, \lambda, \varepsilon)=Y(y, \nu, \varepsilon), \quad x=y+\varepsilon\eta(y, \varepsilon)$$
$$y=x+\varepsilon\xi(x, \varepsilon)=(x+\varepsilon\varphi(x, \varepsilon))(1+\varepsilon\varphi(1, \varepsilon))^{-1}$$
$$\lambda=\nu(1+\varepsilon\varphi(1, \varepsilon))^{-1} \tag{2.5}$$
$$X'=Y'(1+\varepsilon\theta)(1+\varepsilon\varphi_1)^{-1}$$
$$X''=Y''(1+\varepsilon\theta)^2(1+\varepsilon\varphi_1)^{-2}+\varepsilon Y'\theta'(1+\varepsilon\varphi_1)^{-1}$$
$$X'''=Y'''(1+\varepsilon\theta)^3(1+\varepsilon\varphi_1)^{-3}+3\varepsilon Y''\theta'(1+\varepsilon\theta)(1+\varepsilon\varphi_1)^{-2}+ \\ +\varepsilon Y'\theta''(1+\varepsilon\varphi_1)^{-1}$$

Here the primes denote the derivatives with respect to the natural argument: $X' = dX/dx$, $\theta' = d\theta/dx$, $Y' = dY/dy$, etc. Since $y = 0$ for $x = 0$ and $y = 1$ for $x = 1$, the boundary conditions (1.11) for X can be reduced to the corresponding conditions for Y using (2.5):

$$Y=0,\ Y'=0,\ Y''(1+\varepsilon\theta)^2(1+\varepsilon\varphi_1)^{-2}+\varepsilon Y'\theta'(1+\varepsilon\varphi_1)^{-1}=0$$
$$(1+\varepsilon c)[Y'''(1+\varepsilon\theta)^3(1+\varepsilon\varphi_1)^{-3}+3\varepsilon Y''\theta'(1+\varepsilon\theta)(1+\varepsilon\varphi_1)^{-2}+ \\ +\varepsilon Y'\theta''(1+\varepsilon\varphi_1)^{-1}]+\varepsilon c'[Y''(1+\varepsilon\theta)^2(1+\varepsilon\varphi_1)^{-2}+ \\ +\varepsilon Y'\theta'(1+\varepsilon\varphi_1)^{-1}]=0 \tag{2.6}$$
$$x=y=0 \vee x=y=1$$

Hence the perturbed boundary-element problem (2.4)–(2.6) with variable coefficients, which is equivalent to (1.11), has been constructed. First of all, it is required that the general solution of (2.4) be constructed. The application of regular perturbation methods (expansions or successive approximations in powers of the small parameter) does not lead to any secular terms. The perturbed differential equation (2.4) can be replaced by a suitable integro-differential equation. To solve the latter we propose the following recurrent scheme of the method of successive approximations suitable for all real values of ν, $|\nu|<\infty$:

$$Y=Y^{(0)}+\varepsilon L[Y],\ L=I*D,\ Y^{(p+1)}(y, \nu, \varepsilon)= \\ =Y^{(0)}(y, \nu)+\varepsilon L\{Y^{(p)}] \tag{2.7}$$
$$Y^{(0)}(y, \nu)=\sum_{i=0}^{3} c_i\Phi_i(y, \nu), \quad \Phi_{0,2}(y, \nu)=(\operatorname{ch}\nu y\pm\cos\nu y)/(\operatorname{ch}\nu\pm\cos\nu)$$
$$\Phi_{1,3}(y, \nu)=(\operatorname{sh}\nu y\pm\sin\nu y)/(\operatorname{sh}\nu\pm\sin\nu),$$
$$y\in[0, 1],\ p=0, 1, 2, \ldots$$

Here $Y^{(0)}(y, \nu)$ is the known general solution of the unperturbed equation (generating the solution) for (2.4) (for $\varepsilon = 0$), c_i, $i = 0, 1, 2, 3$ being arbitrary constants. Its representation in the form (2.7) is taken to facilitate the passage to the limit as $\nu\to0$ and to ensure boundedness as $\nu\to\infty$. The function $Y^{(0)}$ turns into a polynomial of the third order in y as $\nu\to0$, since $\Phi_i(y, 0) = y^i$. Thus $Y^{(0)}(y, 0) = c_0+c_1y+c_2y^2+c_3y^3$. For all real ν, Φ_i are uniformly bounded, since $0\leqslant y\leqslant1$. Moreover, $\Phi_i\to0$ as $|\nu|\to\infty$ for $0\leqslant y<1$. In (2.7) L is an integro-differential operator, which is a consecutive combination of a third-order differential operator D in y and a Volterra-type integral operator I (in y) with difference kernel. D is defined on the set of functions Y of class C^3:

$$F=F(y, \varepsilon)=D[Y]=D(y, \varepsilon)[Y],\ y\in[0, 1] \tag{2.8}$$
$$D(y, \varepsilon)=A(y, \varepsilon)\frac{d^3}{dy^3}+B(y, \varepsilon)\frac{d^2}{dy^2}+C(y, \varepsilon)\frac{d}{dy}$$

The elements F form a set of continuous functions to which to apply the integral operator I:

$$Z=I[F]=I(y,\nu)[F]=\int_0^y G(y-z,\nu)F(z,\varepsilon)\,dz \tag{2.9}$$

$$G(y,\nu)=(\operatorname{sh}\nu y-\sin\nu y)/2\nu^3,\ y\in[0,1],\ |\nu|<\infty,\ G\approx y^3/3!,\ (y\nu)\to 0$$

$$G(0,\nu)=G'(0,\nu)=G''(0,\nu)=0,\ G'''(0,\nu)=1$$

The elements of Z form a set of functions of class C^4. The difference kernel G has the fourth-order smoothing property. Since Φ_i are analytic functions of y for all real ν such that $|\nu|<\infty$, the recurrent scheme (2.7) is well defined. We observe that the application of D to $Y^{(0)}(y,\nu)$, $Y^{(1)}(y,\nu,\varepsilon),\ldots,Y^{(p)}(y,\nu,\varepsilon),\ldots$ leads to multipliers of order ν^3 as $|\nu|\to\infty$, which are cancelled when the operator I ($\|I\|\sim|\operatorname{sh}\nu y|\nu^{-3}$) is applied. It is important to note that, according to (2.7), the application of the operator I given by (2.9) does not lead to an exponential increase in $Y^{(p)}(y,\nu,\varepsilon)$ as $|\nu|\to\infty$, i.e. the functions $Y^{(p)}$ turn out to be bounded for $y\in[0,1]$ and $0\leqslant\varepsilon\leqslant\varepsilon_0$ uniformly with respect to ν such that $|\nu|<\infty$, $\varepsilon_0>0$ being sufficiently small. Indeed, at the pth step the main term in the integrand of the exponential asymptotic form with respect to ν has the form

$$|\operatorname{sh}\nu(y-z_0)\operatorname{sh}\nu(z_0-z_1)\ldots\operatorname{sh}\nu(z_{q-1}-z_q)\operatorname{ch}\nu z_q/\operatorname{ch}\nu| \tag{2.10}$$

$$1\geqslant y\geqslant z_0\geqslant z_1\geqslant\ldots\geqslant z_{q-1}\geqslant z_q\geqslant 0,\quad q=0,1,\ldots,p$$

or a form similar to this expression. The factors multiplying this term are uniformly bounded functions of $z_0, z_1,\ldots,z_p$ and ν, ε. Analysis of the exponents of the expressions of the type (2.10) leads to the following quantities:

$$|\pm\nu(y-z_0)\pm\nu(z_0-z_1)\pm\ldots\pm\nu(z_{q-1}-z_q)\pm\nu z_q|-|\nu|$$

Here the signs $\pm$ in front of each term are independent. As a result,we obtain 2^{q+1} expressions for the first term under the modulus sign. The maximum value $|\nu|y$ of the first modulus is attained for $z_k=z_{k-1}$. Hence we find that $|\nu|y-|\nu|\leqslant 0$ for $0\leqslant y\leqslant 1$, i.e. $Y^{(p)}\to 0$ exponentially with respect to $|\nu|$ for $y<1$, $|Y^{(p)}|\sim\exp(-|\nu|(1-y))$, and $Y=O(1)$ as $\nu\to\infty$ for $y=1$ only.

Consider the recurrent procedure (2.7). By the linearity of L, we have

$$Y^{(p+1)}=Y^{(0)}+\sum_{i=1}^{p}\varepsilon^iL^i[Y^{(0)}],\quad L^{i+1}[Y]\equiv L[L^i[Y]],\quad L^0=E,\ L^0[Y]\equiv Y \tag{2.11}$$

For sufficiently small $\varepsilon<0$, εL is a contracting operator. On the basis of Banach's theorem [6] it can be established that Eq. (2.7) has a unique solution, which can be obtained as the limit of the sequence (2.11):

$$\lim_{p\to\infty}Y^{(p+1)}=Y^*=Y^{(0)}+\sum_{i=1}^{\infty}\varepsilon^iL^i[Y^{(0)}]=(E-\varepsilon L)^{-1}[Y^{(0)}] \tag{2.12}$$

For $|\varepsilon|<\varepsilon_0$, with $\varepsilon_0=\|L\|^{-1}$, where the norm $\|L\|$ of the bounded operator L can be constructively expressed in terms of the coefficients A, B, C, and the kernel G, the operator εL is contracting and the successive approximations (2.11) converge uniformly to the desired solution (2.12) of (2.7) and (2.4).

We remark that if θ and c are quadruply differentiable (see expressions (2.4) for A, B and C), then, using the integration-by-parts formula and the properties of the kernel G in (2.9), one can reduce (2.7) to an integral equation, whose terms, however, do not satisfy the uniform boundedness condition with respect to ν. In the first-order approximation with respect to ε [with an error $O(\varepsilon^2)$], when constructing the desired solution Y, it suffices oneself to expression (2.8) for D for $\varepsilon=0$, i.e. to take

$$D(y,0)[Y^{(0)}]=[-2(3\theta'+c')Y^{(0)'''}-\\ -(4\theta''+c'')Y^{(0)''}-\theta'''Y^{(0)'}]|_{x=y} \tag{2.13}$$

where $Y^{(0)}$ is a function known to within the choice of the coefficients c_i. Thus we obtain the explicit expressions

$$Y^{(p)}(y,\nu,\varepsilon)=\sum_{i=0}^{3}c_i\Phi_i^{(p)}(y,\nu),\quad Y^*(y,\nu,\varepsilon)=\sum_{i=0}^{3}c_i\Phi_i^*(y,\nu) \tag{2.14}$$

$$\Phi_i^*(y,\nu,\varepsilon)=\lim_{p\to\infty}\Phi_i^{(p)}(y,\nu,\varepsilon),\quad \Phi_i^{(p)}(y,\nu,0)=\Phi_i^*(y,\nu,0)=\Phi_i(y,\nu)$$

$$Y^{(p)}(y,\nu,0)=Y^*(y,\nu,0)=Y^{(0)}(y,\nu)$$

for the desired functions $Y^{(p)}(y,\nu,\varepsilon)$ for each step p and the limiting function $Y^*(y,\nu,\varepsilon)$.

We remark that the functions $Y^{(p)}$ and Y^* are differentiable with respect to ν and ε. The differentiation with respect to y leads to a factor $O(\nu^k)$, where k is the order of the derivative, since the dependence on y is realized as the product νy.

Since the operators D and L are analytical with respect to ε for $|\varepsilon|\leqslant\varepsilon_0$, the desired solution $Y^{(p)}$ [the approximation with error $O(\varepsilon^{p+1})$] or Y^* (the limit as $p\to\infty$) can be represented in the form of a finite sum or a uniformly convergent series in terms of the powers of ε, respectively:

$$Y^{(p)}=Y^{(0)}+\sum_{l=1}^{p}\varepsilon^lY_l,\quad (p\geqslant1),\quad Y^*=Y^{(0)}+\sum_{l=1}^{\infty}\varepsilon^lY_l \tag{2.15}$$

$$Y_1=L_1Y^{(0)},\ Y_2=L_2Y^{(0)}+L_1Y_1,\dots,\ Y_p=L_pY^{(0)}+L_{p-1}Y_1+\dots+L_1Y_{p-1}$$

$$\varepsilon L(y,\nu,\varepsilon)=\sum_{l=1}^{\infty}\varepsilon^lL_l(y,\nu)$$

$$\varepsilon D(y,\varepsilon)=\sum_{l=1}^{\infty}\varepsilon^lD_l(y)=\sum_{l=1}^{\infty}\varepsilon^l\left[A_l(y)\frac{d^3}{dy^3}+B_l(y)\frac{d^2}{dy^2}+C_l(y)\frac{d}{dy}\right]$$

Here $A_l(y)$, $B_l(y)$ and $C_l(y)$ are the coefficients of the Taylor expansions of $A(y,\varepsilon)$, $B(y,\varepsilon)$ and $C(y,\varepsilon)$ in ε. Analytic computations based on (2.15) may turn out to be much simpler than those carried out using the recurrent scheme (2.7) or (2.11), which is usually more suitable for numerical computations in specific cases. The substitution of $Y^{(p)}$ and Y^* into the boundary conditions (2.6), taking (1.11) into account leads to a transcendental secular (characteristic) equation for the unknown parameter ν:

$$\Delta^{(p)}(\nu,\varepsilon)\equiv\Delta^{(0)}(\nu)+\varepsilon\Gamma^{(p-1)}(\nu,\varepsilon)=0,\quad \Gamma^{(-1)}\equiv0 \tag{2.16}$$

$$\Delta^*(\nu,\varepsilon)\equiv\Delta^{(0)}(\nu)+\varepsilon\Gamma^*(\nu,\varepsilon)=0,\quad \Delta^*=\lim_{p\to\infty}\Delta^{(p)}$$

The desired approximate and limiting ("exact") solutions of (2.16) can be constructed by means of a recurrence procedure of the form

$$\Delta^{(0)}(\nu_n^{(l)})=-\varepsilon\Gamma^{(p-1)}(\nu_n^{(l-1)},\varepsilon),\quad l=0,\ 1,\dots,p \tag{2.17}$$

$$\Delta^{(0)}(\nu_n^{(l)})=-\varepsilon\Gamma^*(\nu_n^{(l-1)},\varepsilon),\quad l=0,\ 1,\ 2,\dots$$

$$\nu_n^{(0)}=\arg\Delta^{(0)}(\nu),\quad n=0,\ \pm1,\ \pm2,\dots,\quad |\nu_n^{(p)}(\varepsilon)-\nu_n^{(0)}|\leqslant c\varepsilon$$

$$|\nu_n^*(\varepsilon)-\nu_n^{(0)}|\leqslant c\varepsilon,\quad |\nu_n^*(\varepsilon)-\nu_n^{(p)}|\leqslant c\varepsilon^p$$

Here $\{\nu_n^{(0)}\}$ is the denumerable set of eigenvalues of the unperturbed boundary-value problem (for $\varepsilon=0$), which is assumed to be known.

By analogy with $Y^{(p)}$ and Y^*, the expressions for the characteristic determinants $\Delta^{(p)}$ and Δ^* and the eigenvalues $\nu_n^{(p)}$ and ν_n^* of the boundary-value problem can be expressed as sums or series in powers of ε. On determining $\{\Delta^{(p)}(\varepsilon)\}$ and $\{\Delta^*(\varepsilon)\}$ and substituting the results into (2.9) and (2.12), we obtain the desired systems of approximate eigenfunctions $\{Y_n^{(p)}\}=\{X_n^{(p)}\}$ or limiting eigenfunctions $\{Y_n{}^*\} = \{X_n{}^*\}$ $(y = x + \varepsilon\xi)$ of the perturbed boundary value problem. The eigenfunctions have the properties of an orthogonal basis with appropriate weight:

$$(Y_n^{(p)}, Y_m^{(p)})_\mu = (X_n^{(p)}, X_m^{(p)})_\chi = \| Y_n^{(p)} \|_\mu{}^2 \delta_{nm} + O(\varepsilon^{p+1}) = \| X_n^{(p)} \|_\chi{}^2 \delta_{nm} + O(\varepsilon^{p+1}) \tag{2.18}$$

$$(Y_n{}^*, Y_m{}^*)_\mu = (X_n{}^*, X_m{}^*)_\chi = \| Y_n{}^* \|_\mu{}^2 \delta_{nm} = \| X_n{}^2 \|_\chi{}^2 \delta_{nm}$$

$$\mu = \mu(y, \varepsilon) = 1 + \varepsilon\delta(y + \varepsilon\eta(y, \varepsilon)), \quad \chi = \chi(x, \varepsilon) = 1 + \varepsilon\delta(x), \quad \mu dy = \chi dx$$

where $(\cdot, \cdot)_{\mu,\chi}$ denotes the scalar product (the integral with respect to x, $y \in [0, 1]$ with weight μ, χ, respectively), and where $\|\cdot\|_{\mu,\chi}$ are the weighted norms. We remark that in the case of a free rod, i.e. in the case of boundary condition 2 at both ends in (1.11), zero is a double eigenvalue ($\lambda = \nu = 0$) with two corresponding (non-orthogonal) eigenfunctions, which can be orthogonalized with weight χ or μ, respectively.

3. SOLUTION OF THE PROBLEM OF THE CONTROLLED MOTION OF AN ELASTIC ROD

On the basis of the systems of eigenvalues $\{\lambda_n(\varepsilon)\}$, $\lambda_n = \nu_n/(1 + \varepsilon\varphi_1)$ and orthonormalized eigenfunctions $\{X_n(x, \varepsilon)\}$, $X_n(x, \varepsilon) \equiv Y_n(y(x, \varepsilon), \varepsilon)$ constructed with the given accuracy in terms of the small parameter ε, the solutions of the boundary-value problems (1.1), (1.3)–(1.6) with the initial conditions (1.7) can be reduced, using the Fourier method and Hilbert's approach [11], to a denumerable system of ordinary differential equations

$$u(t, x) = \sum_{n=-\infty}^{\infty} \theta_n(t) X_n(x), \quad \theta_n{}^{\cdot\cdot} + \lambda_n{}^4\theta_n = W_n(t) + [-((1 + \varepsilon c) u'')' X_n +$$

$$+ (1 + \varepsilon c)(u''X' - u'X'') + u((1 + \varepsilon c) X'')']_{x=0}^{x=1} \equiv Q_n(t) \tag{3.1}$$

$$\theta_n = (u, X_n)_\chi, \quad W_n = (W, X_n)_\chi$$

$$\theta_n(0) = f_n{}^0 = (f^0, X_n)_\chi, \quad \theta_n{}^\cdot(0) = g_n{}^0 = (g^0, X_n)_\chi$$

(the dependence of u, X_n, θ_n, and other functions on ε is suppressed to simplify the notation). Taking into account the boundary conditions (1.3)–(1.6) for $u(t, x)$ and (1.11) for $X_n(x)$ for the given functions $S_{0,1}(t)$, $K_{0,1}(t)$, $M_{0,1}(t)$ and $P_{0,1}(t)$, we obtain 10 forms of the known right-hand sides $Q_n(t) = Q_n{}^{(j,i)}(t) = Q_n{}^{(i,j)}(t)$ $(i, j = 1, 2, 3, 4)$ for the denumerable system (3.1). Each of the initial-value problems has the elementary solution

$$\theta_n^{(i,j)} = f_n{}^0 \cos \lambda_n{}^2 t + g_n{}^0 \lambda_n^{-2} \sin \lambda_n{}^2 t + \lambda_n^{-2} \int_0^t \sin \lambda_n{}^2 (t - \tau) Q_n^{(i,j)}(\tau)\, d\tau \tag{3.2}$$

In the case of the control problem, i.e. in the case when the conditions (1.8) are given for $t = T$, the control functions $Q_n{}^{(i,j)}$ from an admissible class must be chosen in such a way that $\theta_n(T) = f_n{}^T$, $\theta_n{}^\cdot(T) = g_n{}^T$. The problems concerned with the existence of a solution of the control problem and the construction of the solution are rather difficult and merit a separate study. There is extensive literature devoted to these problems (see, for example, [12–19] and other references).

4. FINDING THE EIGENFUNCTIONS AND EIGENVALUES IN THE FIRST APPROXIMATION FOR AN INHOMOGENEOUS ROD WITH HINGED SUPPORT

We assume that the rod has a circular or rectangular cross-section (see the Fig. 1). Let r_0 be the inner radius and let r_1 be the outer radius, or let a_0 and b_0 be the dimensions of the inner rectangle and let a_1 and b_1 be the dimensions of the outer rectangle. If the volume density ρ_V and Young's modulus E are constant, we obtain the corresponding expressions [1, 2, 5]

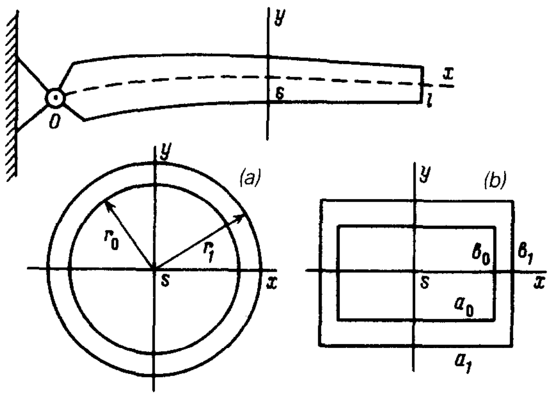

FIG. 1.

$$\rho(x)=\rho_V S(x),\quad \sigma(x)=EI(x) \tag{4.1}$$

for the linear density and bending stiffness, S being the surface area of the cross-section at x and I the moment of inertia about one of the main axes. In the case of a circular cross-section (Fig. 1a), we obtain

$$S(x)=\pi(r_1^2(x)-r_0^2(x)),\quad I(x)={}^1/_4\pi(r_1^4(x)-r_0^4(x)) \tag{4.2}$$

For a rectangular cross-section (Fig. 1b), we have

$$S(x)=a_1(x)b_1(x)-a_0(x)b_0(x),$$
$$I_y(x)=(a_1(x)b_1^3(x)-a_0(x)b_0^3(x))/12 \tag{4.3}$$

(I_y is the moment about the y-axis).

If we restrict ourselves only to the linear terms in the expansion in powers of ε (in the case of a small variation of the parameters) and assume that the dimensions of the cross-section of the rod depend linearly on x, then, on the basis of (4.1)–(4.3), we obtain the following approximate expressions for the circular and rectangular cross-sections, respectively:

$$\rho(x)=\rho_0(1+\varepsilon\delta x)+O(\varepsilon^2),\quad \sigma(x)=\sigma_0(1+\varepsilon cx)+O(\varepsilon^2) \tag{4.4}$$

$$r_{0,1}(x)=r_{0,1}^0+\varepsilon r_{0,1}^1 x,\quad \rho_0=\pi\rho_V(r_1^{02}-r_0^{02}),\quad \sigma_0={}^1/_4\pi E(r_1^{04}-r_0^{04})$$

$$\delta=2\pi\rho_V(r_1^1 r_1^0-r_0^1 r_0^0)/\rho_0,\quad c=\pi E(r_1^1 r_1^{03}-r_0^1 r_0^{03})/\sigma_0$$

$$a_{0,1}(x)=a_{0,1}^0+\varepsilon a_{0,1}^1 x,\quad b_{0,1}(x)=b_{0,1}^0+\varepsilon b_{0,1}^1 x,\quad \rho_0=\rho_V(a_1^0 b_1^0-a_0^0 b_0^0)$$

$$\sigma_0=E(a_1^0 b_1^{03}-a_0^0 b_0^{03})/12,\quad \delta=\rho_V(a_1^0 b_1^1-a_0^0 b_0^1+a_1^1 b_1^0-a_0^1 b_0^0)/\rho_0$$

$$c=E(3a_1^0 b_1^{02} b_1^1-3a_0^0 b_0^{02} b_0^1+a_1^1 b_1^{03}-a_0^1 b_0^{03})/(12\sigma_0)$$

Equation (1.11) for the eigenfunctions X changes to

$$((1+\varepsilon cx+O(\varepsilon^2))X'')''-\lambda^4(1+\varepsilon\delta x+O(\varepsilon^2))X=0,\quad 0<x<l \tag{4.5}$$

The relations between y and x and between ν and λ take the simple forms

$$y=x+\varepsilon(\delta-c)x(x-1)/8+O(\varepsilon^2) \tag{4.6}$$

$$x=y+\varepsilon(\delta-c)y(1-y)/8+O(\varepsilon^2),\quad \nu=\lambda(1+\varepsilon\varphi_1),\quad \varphi_1=(\delta-c)/8$$

Moreover, we neglect the terms of order ε^2 in the equation for Y.

Having applied the transformations from (2.4), we get

$$Y^{IV}-\nu^4 Y=\varepsilon AY''',\quad A=-(c+3\delta)/2 \tag{4.7}$$

to a first approximation.

We shall find the solution of the perturbed problem in the linear approximation for a weakly inhomogeneous rod with hinged left end and free right end. The boundary conditions for the unperturbed initial-value problem (4.7) can be written as follows:

$$Y|_{y=0}=Y''|_{y=0,1}=Y'''|_{y=1}=0 \tag{4.8}$$

The solution of this problem is known and has the form

$$Y_j^0(y,\nu)=\sum_{i=0}^{3}k_i\Phi_i(y,\nu_j),\quad k_0=k_2=0,\quad k_3=-k_1\frac{\varkappa_+}{\varkappa_-} \tag{4.9}$$

$$\varkappa_\pm=(\operatorname{sh}\nu\pm\sin\nu)(\operatorname{ch}\nu\pm\cos\nu)$$

where Φ_i are functions defined in (2.7) and ν_j can be found from the known transcendental equation

$$\operatorname{sh}\nu\cos\nu=\operatorname{ch}\nu\sin\nu\quad(\operatorname{th}\nu=\operatorname{tg}\nu) \tag{4.10}$$

$$\nu_0=0,\ \nu_{\pm1}=\pm3{,}927,\ \nu_{\pm2}=\pm7{,}069,\dots,\ \nu_{\pm n}=\pm\pi/4\pm\pi n+O(e^{-2|n|\pi})$$

One can prove that the solution of the first approximation of the perturbed problem (4.7) has the form

$$Y(y,\nu)=\sum_{i=0}^{3}k_i^*\Phi_i(y,\nu)\left(1+\frac{\varepsilon}{4}Ay\right) \tag{4.11}$$

for arbitrary boundary conditions.

The boundary conditions (2.6) for the perturbed problem (a hinged attachment) can be written in the form

$$Y|_{y=0}=0,\ [Y''(1+2\varepsilon(\theta-\varphi_1))+\varepsilon\theta'Y']|_{y=0,1}=0 \tag{4.12}$$

$$[Y'''(1+3\varepsilon(\theta-\varphi_1))+3\varepsilon\theta'Y'']|_{y=1}=0$$

where $\theta=(\delta-c)/4$, as follows from (2.1). On transforming (4.12) using (2.2), we find that

$$Y|_{y=0}=0,\ [Y''+\varepsilon(\delta-c)(Y'-Y'')/4]|_{y=0}=0 \tag{4.13}$$

$$[Y''+\varepsilon(\delta-c)(Y'+Y'')/4]|_{y=1}=0,$$

$$[Y'''+3\varepsilon(\delta-c)(Y''+Y'''/2)/4]|_{y=1}=0$$

From (4.11) we can derive the following relations for the derivatives of Y with respect to y:

$$Y'=\sum_{i=0}^{3}k_i^*\left(\Phi_i'\left(1+\frac{\varepsilon}{4}Ay\right)+\frac{\varepsilon}{4}A\Phi_i\right),$$

$$Y''=\sum_{i=0}^{3}k_i^*\left(\Phi_i''\left(1+\frac{\varepsilon}{4}Ay\right)+\frac{\varepsilon}{2}A\Phi_i'\right)$$

$$Y'''=\sum_{i=0}^{3}k_i^*\left(\Phi_i'''\left(1+\frac{\varepsilon}{4}Ay\right)+\frac{3\varepsilon}{4}A\Phi_i''\right) \tag{4.14}$$

From the first condition in (4.13) it follows that $k_0^*=0$. Substituting the values of Y and its derivatives from (4.11) and (4.14) into (4.13) and using the values of Φ_i for $y=0$ and $y=1$, after some algebra we obtain the relations

$$k_2^*=\varepsilon k_1^*(\delta+c)/(2\Phi_2'(1,\nu)) \tag{4.15}$$

$$k_3^*=-k_1^*(\Phi_1''/\Phi_3'')(1+\varepsilon(\delta+c)/(2\Phi_2'))|_{y=1}=0$$

expressing k_2^* and k_3^* in terms of k_1^* as well as the transcendental characteristic equation

$$[\Phi_1'''-\Phi_1''\Phi_3'''/\Phi_3''+\varepsilon(\delta+c)(\Phi_2'''-\Phi_1''')/(2\Phi_2')]|_{y=1}=0 \tag{4.16}$$

In (4.16) we substitute the values of the derivatives of Φ_i. After a number of trigonometric identity transformations, we get

$$F_0(\nu)+\varepsilon F_1(\nu)=0, \quad F_0(\nu)=\operatorname{ch}\nu\sin\nu-\operatorname{sh}\nu\cos\nu \tag{4.17}$$

$$F_1(\nu)=(\delta+c)(1-\operatorname{ch}\nu\cos\nu)/(2\nu)$$

Since we seek the first approximation of the solution, we have

$$\nu=\nu_0+\varepsilon\nu_1 \tag{4.18}$$

where ν_0 are the eigenvalues of the unperturbed problem obtained from (4.9) and ν_1 can be found from (2.17):

$$\nu_1=-F_1(\nu_0)/F'_{0,\nu}(\nu_0), \quad F'_{0,\nu}(\nu_0)=2\operatorname{sh}\nu_0\sin\nu_0 \tag{4.19}$$

Substituting ν_1 from (4.18) and (4.19), $k_2{}^*$ and $k_3{}^*$ from (4.15), and $k_0{}^*=0$ into (4.11), we can find the first approximations of the eigenfunctions Y_l. X_l and λ_l can be determined from (2.2) and (2.3), where

$$\lambda_l=\nu_l(1-\varepsilon\varphi_1)=\nu_{0,l}+\varepsilon(\nu_{1,l}-\nu_{0,l}(\delta-c)/8) \tag{4.20}$$

For example, in the case of a circular cross-section, if $r_0{}^0=r_0{}^1=0$ (a solid rod), then we have $\delta=2r_1{}^1/r_1{}^0$, $c=4r_1{}^1/r_1{}^0$, $(\delta+c)=6r_1{}^1/r_1{}^0$ and $(\delta-c)=-2r_1{}^1/r_1{}^0$. This means that the eigenvalues λ_l increase for $l\gg 1$ ($\nu_l\gg\nu_1\sim 1$ for $l\gg 1$) in the case under consideration. For small l, the dependence is the following:

l	1	2	3	4	5
$\lambda_l{}^0$	3.927	7.069	10.21	13.35	16.49
$\lambda_l{}^1$	1.385	1.978	2.699	3.450	4.214

Here $\varepsilon=0,1$ and $r_1{}^1/r_1{}^0=1$.

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